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THE FUNDAMENTAL SOLUTION FOR THE THEORY OF ORTHOTROPIC SHALLOW SHELLS INVOLVING SHEAR DEFORMATION

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Abstract—A kind of orthotropic shallow shell involving shear deformation is analysed in this paper. The decomposed form of the governing equations of this kind of shell, as well as corresponding fundamental solutions, are obtained. These results have many important applications in theory and engineering problems.

INTRODUCTION

In the development of composite materials, investigations concerning the theory of orthotropic or anisotropic plates and shells remain one of the principal areas of mechanics. For shallow shells, however, many studies are still related to thin shell theory. However, for problems of stress concentration, fracture, medium-thick as well as anisotropic shell structures, effects of transverse shearing deformation have to be considered. In the present paper the theory of orthotropic shallow shells involving shear deformation is discussed in detail. In this kind of problem, investigations are comparatively difficult because of the complexity of the governing equations. In the paper of Delale and Erdogan (1979), a spherical shell with a crack was discussed using the shell theory mentioned above. A more general analysis for orthotropic shallow shells of variable curvature, including transverse shear deformation, is not found in the literature, except when using a finite element approximation (Bernadou, 1993).

In the present paper, Hörmander's operator method is used initially to decompose the governing equations of orthotropic shallow shells involving shear deformation, and a group of decomposed equilibrium equations expressed by displacement functions is obtained. Then, by using a plane-wave decomposition method and some other treatments, the fundamental solution for an orthotropic shallow shell involving shear deformation is obtained. These results are very important for stress and deformation analysis, as well as boundary element analysis of orthotropic shallow shells.

TRANSFORMATION OF BASIC EQUATIONS

Consider an orthotropic shallow shell in which two principle axes coincide with the coordinate axes $0x$ and $0y$, respectively, and with a quadratic middle surface given by

$$z = -1/2(k_1x^2+k_2y^2), \tag{1}$$

where k_1 and k_2 are principle curvatures of the shell in the x- and y-directions, respectively. If the effects of transverse shear deformation are considered further, the basic equations of the shallow shell can be expressed as follows (Delale and Erdogan, 1979):

Strain–displacement relations

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$$\varepsilon_x = \frac{\partial u}{\partial x} + k_1 w, \quad \varepsilon_y = \frac{\partial v}{\partial y} + k_2 w, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \psi_x, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \psi_y,$$

$$\kappa_x = \frac{\partial \psi_x}{\partial x}, \quad \kappa_y = \frac{\partial \psi_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}. \tag{2}$$

Equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = p_x, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = p_y,$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - (k_1 N_x + k_2 N_y) + p_z = 0,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = m_x, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = m_y. \tag{3}$$

Stress–strain relations

$$N_x = B_1(\varepsilon_x + \nu_2 \varepsilon_y), \quad N_y = B_2(\varepsilon_y + \nu_1 \varepsilon_x), \quad N_{xy} = B_k \gamma_{xy},$$

$$Q_x = C_1 \gamma_{xz}, \quad Q_y = C_2 \gamma_{yz}, \quad M_x = D_1(\kappa_x + \nu_2 \kappa_y),$$

$$M_y = D_2(\kappa_y + \nu_1 \kappa_x), \quad M_{xy} = D_k \kappa_{xy}. \tag{4}$$

Here, u, v and w are the displacements in the x, y and z directions, respectively; ψ_x and ψ_y are the normal rotations in the xz and yz planes, respectively, due to bending; N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x and Q_y are components of generalized stresses; p_x, p_y, p_z, m_x and m_y are generalized distributive loads applied in different directions of the shell, respectively; constants B_1, B_2 and B_k are called tension stiffnesses, C_1 and C_2 shear stiffnesses, and D_1, D_2 and D_k bending stiffnesses, respectively, which can be obtained through the constitutive law (see e.g. Lekhnitskii, 1963); ν_1 and ν_2 are Poisson's ratios. For orthotropic materials, the following relations exist among the elastic constants:

$$D_i = \frac{h^2}{12} B_i \quad (i = 1, 2, k), \quad D_1 \nu_2 = D_2 \nu_1, \quad B_1 \nu_2 = B_2 \nu_1, \tag{5}$$

where h is the thickness of the shell. Inserting eqn (2) into eqn (4), the stress–displacement relations can be obtained:

$$N_x = B_1 \left[\frac{\partial u}{\partial x} + \nu_2 \frac{\partial v}{\partial y} + (k_1 + \nu_2 k_2) w \right],$$

$$N_y = B_2 \left[\frac{\partial v}{\partial y} + \nu_1 \frac{\partial u}{\partial x} + (k_2 + \nu_1 k_1) w \right], \quad N_{xy} = B_k \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

$$Q_x = C_1 \left(\frac{\partial w}{\partial x} + \psi_x \right), \quad Q_y = C_2 \left(\frac{\partial w}{\partial y} + \psi_y \right),$$

$$M_x = D_1 \left(\frac{\partial \psi_x}{\partial x} + \nu_2 \frac{\partial \psi_y}{\partial y} \right), \quad M_y = D_2 \left(\frac{\partial \psi_y}{\partial y} + \nu_1 \frac{\partial \psi_x}{\partial x} \right),$$

$$M_{xy} = D_k \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right). \tag{6}$$

Furthermore, substituting eqn (6) into eqn (3), we can obtain the equilibrium equations expressed in terms of displacements:

$$[\mathbf{L}]\{\mathbf{U}\} = \{\mathbf{P}\}, \tag{7}$$

where

$$\{\mathbf{U}\} = [u, v, w, \psi_x, \psi_y]^T, \quad \{\mathbf{P}\} = [p_x, p_y, p_z, m_x, m_y]^T, \tag{8}$$

and [L] is a symmetrical differential operator matrix of order 5 × 5, whose elements are

$$\begin{aligned} &L_{11} = \nabla_{11}^2, \quad L_{12} = A_{12}D_xD_y, \quad L_{13} = A_{13}D_x, \quad L_{14} = L_{15} = 0, \\ &L_{22} = \nabla_{22}^2, \quad L_{23} = A_{23}D_y, \quad L_{24} = L_{25} = 0, \quad L_{33} = \nabla_{33}^2, \\ &L_{34} = -C_1D_x, \quad L_{35} = -C_2D_y, \quad L_{44} = \nabla_{44}^2, \quad L_{45} = A_{45}D_xD_y, \\ &L_{55} = \nabla_{55}^2, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &A_{12} = B_1v_2 + B_k, \quad A_{13} = B_1(k_1 + v_2k_2), \quad A_{23} = B_2(k_2 + v_1k_1), \\ &A_{33} = B_1k_1^2 + B_2k_2^2 + 2B_1v_2k_1k_2, \quad A_{45} = D_1v_2 + D_k, \end{aligned} \tag{10}$$

and D_x, D_y, ∇_{ii}^2 ($i = 1, 2, \ldots, 5$) are partial differential operators given by

$$\begin{aligned} &D_x = \frac{\partial}{\partial x}, \quad D_y = \frac{\partial}{\partial y}, \quad \nabla_{11}^2 = B_1D_x^2 + B_kD_y^2, \\ &\nabla_{22}^2 = B_kD_x^2 + B_2D_y^2, \quad \nabla_{33}^2 = A_{33} - (C_1D_x^2 + C_2D_y^2), \\ &\nabla_{44}^2 = D_1D_x^2 + D_kD_y^2 - C_1, \quad \nabla_{55}^2 = D_kD_x^2 + D_2D_y^2 - C_2. \end{aligned} \tag{11}$$

Since the partial differential equations (7) are coupled, it is very difficult to solve them directly. By using Hörmander's operator method (Hörmander, 1963), eqns (7) can be decomposed. For this reason, we define the following displacement functions:

$$\{\mathbf{\Phi}\} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5]^T, \tag{12}$$

which satisfy the relation

$$\{\mathbf{U}\} = [\mathbf{F}]\{\mathbf{\Phi}\}, \tag{13}$$

where the symmetric differential operator matrix [F] is the accompanying matrix of [L], whose elements are

$$\begin{aligned} &F_{11} = \nabla_{22}^2J_8 - A_{23}^2J_4D_y^2, \quad F_{12} = (A_{13}A_{23}J_4 - A_{12}J_8)D_xD_y, \\ &F_{13} = J_3J_4, \quad F_{14} = -J_3J_6, \quad F_{15} = J_3J_5, \quad F_{22} = \nabla_{11}^2J_8 - A_{13}^2J_4D_x^2, \\ &F_{23} = -J_2J_4, \quad F_{24} = J_2J_6, \quad F_{25} = -J_2J_5, \quad F_{33} = J_1J_4, \\ &F_{34} = -J_1J_6, \quad F_{35} = J_1J_5, \quad F_{44} = \nabla_{55}^2J_7 - C_2^2J_1D_x^2, \\ &F_{45} = (C_1C_2J_1 - A_{45}J_7)D_xD_y, \quad F_{55} = \nabla_{44}^2J_7 - C_1^2J_1D_x^2, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &J_1 = \nabla_{11}^2\nabla_{22}^2 - A_{12}^2D_x^2D_y^2, \quad J_2 = A_{23}\nabla_{11}^2D_y - A_{12}A_{13}D_x^2D_y, \\ &J_3 = A_{12}A_{13}D_xD_y^2 - A_{13}\nabla_{22}^2D_x, \quad J_4 = \nabla_{44}^2\nabla_{55}^2 - A_{45}^2D_x^2D_y^2, \end{aligned}$$

$$J_5 = C_2\nabla_{44}^2 D_y - C_1 A_{45} D_x^2 D_y, \quad J_6 = C_2 A_{45} D_x D_y^2 - C_1 \nabla_{55}^2 D_x,$$

$$J_7 = \nabla_{11}^2\nabla_{22}^2\nabla_{33}^2 + 2A_{12}A_{13}A_{23}D_x^2D_y^2 - A_{23}^2\nabla_{11}^2 D_y^2 - A_{13}\nabla_{22}^2 D_x^2 - A_{12}^2\nabla_{33}^2 D_x^2 D_y^2,$$

$$J_8 = \nabla_{33}^2\nabla_{44}^2\nabla_{55}^2 + 2C_1C_2A_{45}D_x^2D_y^2 - A_{45}^2\nabla_{33}^2 D_x^2 D_y^2 - C_2^2\nabla_{44}^2 D_y^2 - C_1^2\nabla_{55}^2 D_x^2. \tag{15}$$

Substituting eqn (13) into (7) and noting the definition of matrix [**F**], we have

$$\mathscr{L}\{\mathbf{\Phi}\} = \{\mathbf{P}\}, \tag{16}$$

where $\mathscr{L}$ is a differential operator given by the determinant of the operator matrix [**L**] and given by

$$\mathscr{L} = J_4 J_7 + J_1(C_1 J_6 D_x - C_2 J_5 D_y). \tag{17}$$

In this way, the coupled equations (7) have now been reduced to a set of uncoupled equations for five displacement functions. Let $\Phi(x,y)$ be the fundamental solution of the differential operator $\mathscr{L}$, i.e.

$$\mathscr{L}(\Phi) = \delta(x,y), \tag{18}$$

where $\delta(x,y)$ is the Dirac δ-function. Then the particular solutions to eqns (16) can be expressed in the form

$$\Phi_j(x,y) = \iint \Phi(x-\zeta, y-\eta) p_j \, d\zeta \, d\eta \quad (j = 1,2,\ldots,5). \tag{19}$$

Once $\mathbf{\Phi}_j(x,y)$ is known, the displacements can be determined from eqn (13) and the generalized stress resultants can be obtained from relations (6):

$$\{\mathbf{T}\} = [\mathbf{R}]\{\mathbf{\Phi}\}, \tag{20}$$

where

$$\{\mathbf{T}\} = [N_x, N_y, N_{xy}, Q_x, Q_y, M_x, M_y, M_{xy}]^T, \tag{21}$$

and the elements of the differential operator matrix [**R**] are listed in Appendix 1.

Suppose a set of concentrated load components is acting on a point (x_0, y_0) of the shell in different directions:

$$p_j = P_j \delta(x - x_0, y - y_0) \quad (j = 1,2,\ldots,5). \tag{22}$$

Inserting eqn (22) into (19) yields

$$\Phi_j = P_j \Phi(x - x_0, y - y_0) \quad (j = 1,2,\ldots,5). \tag{23}$$

Substituting the above expressions into eqns (13) and (20), we can therefore take the stress–strain analysis for orthotropic shallow shell members acted upon by concentrated forces. Moreover, the fundamental solution $\Phi(x,y)$ can also be used to construct the kernels of boundary integral equations for BEM analysis (Lu and Huang, 1992).

FUNDAMENTAL SOLUTION $\Phi(x,y)$

The analysis of the above section reduces the effort to find the fundamental solution $\Phi(x,y)$ restricted by eqn (18), i.e.

$$[J_4J_7+J_1(C_1J_6D_x-C_2J_5D_y)]\Phi(x,y) = \delta(x,y). \tag{24}$$

This is a tenth order partial differential equation and can be transformed to an ordinary differential equation to be solved by using the plane-wave decomposition method (Gel'fand and Shilov, 1966).

For two-dimensional problems, let

$$\rho = \omega_1 x+\omega_2 y, \tag{25}$$

where (ω_1, ω_2) are the coordinates of a point on the unit circle:

$$\omega_1 = \cos\theta, \quad \omega_2 = \sin\theta. \tag{26}$$

Therefore, $\delta(x,y)$ can be expressed as (Gel'fand and Shilov, 1966):

$$\delta(x,y) = -\frac{1}{4\pi^2}\int_0^{2\pi} |\rho|^{-2}\,d\theta. \tag{27}$$

The fundamental solution $\Phi(x,y)$, written in the following form,

$$\Phi(x,y) = \int_0^{2\pi} \varphi(\rho)\,d\rho, \tag{28}$$

is called the plane-wave representation of the fundamental solution. Substituting eqns (27) and (28) into eqn (24), and taking note of the following relations:

$$\frac{\partial}{\partial x} = \omega_1\frac{\partial}{\partial\rho}, \quad \frac{\partial}{\partial y} = \omega_2\frac{\partial}{\partial\rho}, \tag{29}$$

we can obtain, after proper simplifications, the following tenth order ordinary differential equation:

$$\frac{d^4}{d\rho^4}\left[a_0\frac{d^6}{d\rho^6}+a_1\frac{d^4}{d\rho^4}+a_2\frac{d^2}{d\rho^2}+a_3\right]\varphi(\rho) = -\frac{1}{4\pi^2}|\rho|^{-2}, \tag{30}$$

where

$$\begin{aligned} a_0 &= -K_3Q_1Q_2, \quad a_1 = Q_2(A_{33}Q_1+Q_7)+Q_1(K_3Q_3+Q_6), \\ a_2 &= -Q_3(A_{33}Q_1+Q_7), \quad a_3 = C_1C_2(A_{33}Q_1+Q_7); \end{aligned} \tag{31}$$

the coefficients K_j $(j = 1,2,\ldots,5)$, Q_j $(j = 1,2,\ldots,9)$ are listed in Appendix 1. After four integrations for eqn (30), we have

$$\left[a_0\frac{d^6}{d\rho^6}+a_1\frac{d^4}{d\rho^4}+a_2\frac{d^2}{d\rho^2}+a_3\right]\varphi(\rho) = \frac{1}{8\pi^2}\rho^2\ln|\rho|. \tag{32}$$

The above equation can be further written as

$$\left(\frac{d^2}{d\rho^2}-r_1^2\right)\left(\frac{d^2}{d\rho^2}-r_2^2\right)\left(\frac{d^2}{d\rho^2}-r_3^2\right)\varphi(\rho) = \frac{1}{8\pi^2 a_0}\rho^2\ln|\rho|, \tag{33}$$

where r_1^2, r_2^2 and r_3^2 are the roots of the equation

$$r^6+\frac{a_1}{a_0}r^4+\frac{a_2}{a_1}r^2+\frac{a_3}{a_0}=0$$

and are given in Appendix 2. The particular solution of eqn (33) can be written as

$$\varphi(\rho)=\sum_{j=1}^{3}[f_j(\rho)\,e^{r_j\rho}+g_j(\rho)\,e^{-r_j\rho}], \tag{34}$$

in which the functions $f_j(\rho)$ and $g_j(\rho)$ can be obtained by the method of variation of parameters (Ye and Xu, 1978):

$$f_j(\rho)=\tfrac{1}{2}\Lambda_j\left\{\left[r_j^2\rho^2\ln|\rho|+r_j\rho(2\ln|\rho|+1)+(2\ln|\rho|+3)\right]e^{-r_j\rho}+2\int_\rho^\infty\frac{e^{-r_j\sigma}}{\sigma}\,d\sigma\right\},$$

$$g_j(\rho)=\tfrac{1}{2}\Lambda_j\left\{\left[r_j^2\rho^2\ln|\rho|-r_j\rho(2\ln|\rho|+1)+(2\ln|\rho|+3)\right]e^{r_j\rho}-2\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}\,d\sigma\right\}$$

$$(j=1,2,3), \tag{35}$$

where

$$\Lambda_j=-\frac{1}{8\pi^2a_0r_j^4\prod_{i=1,i\neq j}^{3}(r_j^2-r_i^2)}\quad (j=1,2,3). \tag{36}$$

Substituting eqn (35) into (34) and after proper simplifications we have

$$\varphi(\rho)=\rho^2\ln|\rho|\sum_{j=1}^{3}r_j^2\Lambda_j+(2\ln|\rho|+3)\sum_{j=1}^{3}\Lambda_j+\sum_{j=1}^{3}\Lambda_j\left[e^{r_j\rho}\int_\rho^\infty\frac{e^{-r_j\sigma}}{\sigma}\,d\sigma-e^{-r_j\rho}\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}\,d\sigma\right]. \tag{37}$$

The integrals in the above equation can be expressed by an exponential integral $E_1(z)$ (Abramowitz and Stegun, 1966)

$$\int_\rho^\infty\frac{e^{-r_j\sigma}}{\sigma}\,d\sigma=E_1(r_j\rho)+\frac{\pi i}{2}\left\{1-\operatorname{sgn}[\operatorname{Re}(r_j\rho)]\right\}\operatorname{sgn}[\operatorname{Im}(r_j\rho)],$$

$$\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}\,d\sigma=-E_1(-r_j\rho)+\frac{\pi i}{2}\left\{1+\operatorname{sgn}[\operatorname{Re}(r_j\rho)]\right\}\operatorname{sgn}[\operatorname{Im}(r_j\rho)], \tag{38}$$

where sgn (x) is the sign function, and Re $(\cdot)$ and Im $(\cdot)$ are the real and imaginary parts of complex variables, respectively. Equation (37) can be rewritten as

$$\varphi(\rho)=\rho^2\ln|\rho|\sum_{j=1}^{3}r_j^2\Lambda_j+(2\ln|\rho|+3)\sum_{j=1}^{3}\Lambda_j+\sum_{j=1}^{3}\Lambda_j\chi_j(\rho), \tag{39}$$

where

$$\chi_j(\rho)=e^{r_j\rho}E_1(r_j\rho)+e^{-r_j\rho}E_1(-r_j\rho)+i\pi\left\{\sinh(r_j\rho)-\cosh(r_j\rho)\operatorname{sgn}[\operatorname{Re}(r_j\rho)]\right\}\operatorname{sgn}[\operatorname{Im}(r_j\rho)]\quad(j=1,2,3). \tag{40}$$

According to the series expansion of the exponential integral (Abramowitz and Stegun, 1966), eqn (39) can also be expressed in the form of a series expansion. Let

$$\alpha_j = \arctan\left|\frac{\mathrm{Re}\,(r_j)}{\mathrm{Im}\,(r_j)}\right| \quad (j = 1,2,3); \tag{41}$$

the arguments of the complex variables $\pm r_j \rho$ can therefore be written as

$$\arg(\pm r_j\rho) = \pm\,\mathrm{sgn}\,[\mathrm{Im}\,(r_j\rho)]\left\{\frac{\pi}{2} \mp \alpha_j\,\mathrm{sgn}\,[\mathrm{Re}\,(r_j\rho)]\right\} \quad (j = 1,2,3). \tag{42}$$

Expanding the exponential integrals and the hyperbolic functions in eqn (40) in series and noting eqn (42), we can obtain after proper treatment

$$\chi_j(\rho) = -2\left\{\gamma + \ln|r_j\rho| + i\left(\frac{\pi}{2} - \alpha_j\right)\mathrm{sgn}\,[\mathrm{Re}\,(r_j\rho)]\,\mathrm{sgn}\,[\mathrm{Im}\,(r_j\rho)]\right\}$$
$$\times \sum_{m=0}^{\infty} \frac{(r_j\rho)^{2m}}{(2m)!} + 2\sum_{m=0}^{\infty} \frac{(r_j\rho)^{2m}}{(2m)!}\psi(2m+1) \quad (j = 1,2,3), \tag{43}$$

where

$$\psi(m+1) = \sum_{s=1}^{m} \frac{1}{s}, \quad \psi(1) = 0, \tag{44}$$

and $\gamma = 0.51721$ is Euler's constant. Substituting eqn (43) into eqn (39) and deleting polynomial terms in ρ and ρ^2, we obtain

$$\varphi(\rho) = -2\sum_{j=1}^{3} \Lambda_j \sum_{m=1}^{\infty} \frac{(r_j\rho)^{2m+2}}{(2m+2)!}\left[\gamma + \ln|r_j\rho| + i\beta_j - \psi(2m+3)\right], \tag{45}$$

where

$$\beta_j = \left(\frac{\pi}{2} - \alpha_j\right)\mathrm{sgn}\,[\mathrm{Re}\,(r_j\rho)]\,\mathrm{sgn}\,[\mathrm{Im}\,(r_j\rho)] \quad (j = 1,2,3). \tag{46}$$

Substituting eqn (39) or eqn (45) into eqn (28), the fundamental solution of eqn (24) can be obtained, in which the definite integral (28) may be solved by numerical integration. Therefore, according to eqn (19), the plane-wave representations of the displacement function are

$$\Phi_j(x,y) = \int_0^{2\pi} \varphi_j(\rho)\,\mathrm{d}\theta = \int_0^{2\pi}\left[\iint \varphi(\tilde{\rho})p_j(\zeta,\eta)\,\mathrm{d}\zeta\,\mathrm{d}\eta\right]\mathrm{d}\theta \quad (j = 1,2,\ldots,5), \tag{47}$$

where $\tilde{\rho} = (x-\zeta)\cos\theta + (y-\eta)\sin\theta$. In particular, when a set of generalized unit concentrated forces is applied to a point of the shallow shell in different directions, the displacement functions are simply $\varphi_j(\rho) = \varphi(\rho)$ $(j = 1,2,\ldots,5)$. Furthermore, according to eqns (13) and (20), the plane-wave representations of the displacements and the generalized stresses can be also expressed as

$$\{\mathbf{U}\} = \int_0^{2\pi} \{\tilde{\mathbf{U}}\}\,\mathrm{d}\theta = \int_0^{2\pi} [\tilde{\mathbf{F}}]\{\varphi\}\,\mathrm{d}\theta, \tag{48}$$

$$\{\mathbf{T}\} = \int_0^{2\pi} \{\tilde{\mathbf{T}}\}\,\mathrm{d}\theta = \int_0^{2\pi} [\tilde{\mathbf{R}}]\{\varphi\}\,\mathrm{d}\theta, \tag{49}$$

where

$$\{\varphi\} = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5]^T, \tag{50}$$

and $[\tilde{\mathbf{F}}]$ and $[\tilde{\mathbf{R}}]$ are differential operator matrices composed of differential operators $\mathrm{d}^k/\mathrm{d}\rho^k$ $(k = 1, 2, \ldots, 9)$; the corresponding elements are given in Appendix 1.

COMPUTATIONAL CONSIDERATIONS

As shown before, it is necessary to compute $\varphi(\rho)$ and its derivatives $(\mathrm{d}^k/\mathrm{d}\rho^k)\varphi(\rho)$ $(k = 1, 2, \ldots, 9)$ during the computation of the displacements and the generalized stresses. We define the function $\lambda_j(\rho)$ as follows:

$$\lambda_j(\rho) = \mathrm{e}^{r_j\rho} E_1(r_j\rho) - \mathrm{e}^{-r_j\rho} E_1(-r_j\rho) + i\pi \left\{ \cosh(r_j\rho) - \sinh(r_j\rho)\,\mathrm{sgn}\,[\mathrm{Re}\,(r_j\rho)] \right\} \mathrm{sgn}\,[\mathrm{Im}\,(r_j\rho)] \quad (j = 1, 2, 3). \tag{51}$$

According to the derivative property of the exponential integral (Abramowitz and Stegun, 1966), we have

$$\frac{\mathrm{d}\lambda_j(\rho)}{\mathrm{d}\rho} = r_j \chi_j(\rho) \quad (j = 1, 2, 3), \tag{52}$$

where $\chi_j(\rho)$ is given by eqn (40), and the following relation can also be obtained:

$$\frac{\mathrm{d}\chi_j(\rho)}{\mathrm{d}\rho} = r_j \lambda_j(\rho) - \frac{2}{\rho} \quad (j = 1, 2, 3). \tag{53}$$

Moreover, by using the relations of eqn (36) and Appendix 2, we can obtain

$$\begin{gathered} \sum_{j=1}^{3} \Lambda_j = -\frac{1}{8\pi^2} \frac{a_2}{a_3^2}, \quad \sum_{j=1}^{3} r_j^2 \Lambda_j = \frac{1}{8\pi^2} \frac{1}{a_3}, \\ \sum_{j=1}^{3} r_j^4 \Lambda_j = \sum_{j=1}^{3} r_j^6 \Lambda_j = 0, \quad \sum_{j=1}^{3} r_j^8 \Lambda_j = -\frac{1}{8\pi^2} \frac{1}{a_0}, \end{gathered} \tag{54}$$

where a_0, a_2 and a_3 are given by eqn (31). Therefore, combining eqns (39), (40) and (51)–(54), we have

$$\begin{aligned} \varphi(\rho) &= (2 \ln|\rho| + 3) \sum_{j=1}^{3} \Lambda_j + \rho^2 \ln|\rho| \sum_{j=1}^{3} r_j^2 \Lambda_j + \sum_{j=1}^{3} \Lambda_j \chi_j(\rho), \\ \frac{\mathrm{d}\varphi}{\mathrm{d}\rho} &= \rho(2 \ln|\rho| + 1) \sum_{j=1}^{3} r_j^2 \Lambda_j + \sum_{j=1}^{3} \Lambda_j r_j \lambda_j(\rho), \\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}\rho^2} &= (2 \ln|\rho| + 3) \sum_{j=1}^{3} r_j^2 \Lambda_j + \sum_{j=1}^{3} \Lambda_j r_j^2 \chi_j(\rho), \\ \frac{\mathrm{d}^{2k+1}\varphi}{\mathrm{d}\rho^{2k+1}} &= \sum_{j=1}^{3} \Lambda_j r_j^{2k+1} \lambda_j(\rho) \quad (k = 1, 2, 3), \\ \frac{\mathrm{d}^{2(k+1)}\varphi}{\mathrm{d}\rho^{2(k+1)}} &= \sum_{j=1}^{3} \Lambda_j r_j^{2(k+1)} \chi_j(\rho) \quad (k = 1, 2, 3), \\ \frac{\mathrm{d}^9\varphi}{\mathrm{d}\rho^9} &= \sum_{j=1}^{3} \Lambda_j r_j^9 \lambda_j(\rho) + \frac{1}{4\pi^2 a_0} \frac{1}{\rho}. \end{aligned} \tag{55}$$

Now, the problem of calculating the derivative values of the fundamental solutions is reduced to evaluating the values of χ_j and λ_j ($j = 1,2,3$), which is easier from the computational point of view.

The series form of the function $\chi_j(\rho)$ can be expressed according to eqns (43) and (46) as

$$\chi_j(\rho) = -2\sum_{m=0}^{\infty}\frac{(r_j\rho)^{2m}}{(2m)!}\left[\gamma+\ln|r_j\rho|+i\beta_j-\psi(2m+1)\right] \quad (j = 1,2,3). \tag{56}$$

Substituting the above equation into eqn (53), after proper arrangements, we have

$$\lambda_j(\rho) = -2\sum_{m=1}^{\infty}\frac{(r_j\rho)^{2m-1}}{(2m-1)!}\left[\gamma+\ln|r_j\rho|+i\beta_j-\psi(2m)\right] \quad (j = 1,2,3). \tag{57}$$

If $|r_j\rho| < 1$, the convergence rate is fast by using eqns (56) and (57) for the evaluation of χ_j and λ_j. If $|r_j\rho|$ is large, eqns (40) and (51) have to be used to compute the functions χ_j and λ_j. In this case, the problem of numerical computation for the exponential integrals $E_1(r_j\rho)$ and $E_1(-r_j\rho)$ will be encountered, which has been discussed in detail by Lu and Huang (1991).

To calculate displacements and generalized stresses, the following integrals will be treated:

$$\frac{\partial^{k+l}\Phi(x,y)}{\partial x^k\,\partial y^l} = \int_0^{2\pi}\cos^k\theta\sin^l\theta\frac{\mathrm{d}^{k+l}\varphi(\rho)}{\mathrm{d}\rho^{k+l}}\,\mathrm{d}\theta \quad (k,l = 0,1,2,\ldots). \tag{58}$$

This can be calculated by sub-region Gaussian numerical integration, in which the values of $\varphi(\rho)$ and its derivatives can be obtained in eqn (55). Detailed treatments are given by Lu and Huang (1991). Numerical examples and the extension to boundary element analysis will be given in another paper.

CONCLUSIONS

In the present paper, a set of uncoupled equilibrium equations for orthotropic shallow shells involving shear deformation is derived. The fundamental solution for the shells is obtained, and corresponding computational formulations are given. The method can be used in stress–strain analyses for composite, rib-shell structures etc. The fundamental solutions derived in this paper can also be used to construct the kernels of boundary integral equations, which are essential work for BEM analysis.

In addition, the treatments for decomposing governing equations with Hörmander's operator method and constructing fundamental solutions with the plane-wave decomposition method, which are used in this paper, can also be extended to analyses of anisotropic plate and shell structures. It is an effective method of treating complex mechanics problems.

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APPENDIX 1

The elements of differential operator matrix [**R**] are

$$R_{1j} = B_1[F_{1j}D_x + v_2F_{2j}D_y + (K_1 + v_2k_2)F_{3j}],$$
$$R_{2j} = B_2[v_1F_{1j}D_x + F_{2j}D_y + (K_2 + v_1k_1)F_{3j}],$$
$$R_{3j} = B_k[F_{1j}D_y + F_{2j}D_x], \quad R_{4j} = C_1[F_{3j}D_x + F_{4j}],$$
$$R_{5j} = C_2[F_{3j}D_y + F_{5j}], \quad R_{6j} = D_1[F_{4j}D_x + v_2F_{5j}D_y],$$
$$R_{7j} = D_2[v_1F_{4j}D_x + F_{5j}D_y], \quad R_{8j} = D_k[F_{4j}D_y + F_{5j}D_x] \quad (j = 1, 2, \ldots, 5),$$

where the differential operators F_{ij} are given by eqn (14).

The coefficients K_j and Q_j are defined as

$$K_1 = B_1\omega_1^2 + B_k\omega_2^2, \quad K_2 = B_k\omega_1^2 + B_2\omega_2^2, \quad K_3 = C_1\omega_1^2 + C_k\omega_2^2,$$
$$K_4 = D_1\omega_1^2 + D_k\omega_2^2, \quad K_5 = D_k\omega_1^2 + D_2\omega_2^2,$$
$$Q_1 = K_1K_2 - A_{12}^2\omega_1^2\omega_2^2, \quad Q_2 = K_4K_5 - A_{45}^2\omega_1^2\omega_2^2, \quad Q_3 = C_2K_4 + C_1K_5,$$
$$Q_4 = C_2K_4 - C_1A_{45}\omega_1^2, \quad Q_5 = C_2A_{45}\omega_2^2 - C_1K_5,$$
$$Q_6 = 2C_1C_2A_{45}\omega_1^2\omega_2^2 - C_2^2K_4\omega_2^2 - C_1^2K_5\omega_1^2,$$
$$Q_7 = 2A_{12}A_{13}A_{23}\omega_1^2\omega_2^2 - A_{23}^2K_1\omega_2^2 - A_{13}^2K_2\omega_1^2,$$
$$Q_8 = A_{23}K_1 - A_{12}A_{13}\omega_1^2, \quad Q_9 = A_{12}A_{13}\omega_2^2 - A_{13}K_2,$$

where A_{11}, A_{13}, A_{33} and A_{45} are given by eqn (10). Let

$$D_\rho = \frac{\mathrm{d}}{\mathrm{d}\rho}, \quad D_\rho^k = \frac{\mathrm{d}^k}{\mathrm{d}\rho^k}.$$

Therefore, the elements of differential operator matrices [$\tilde{\mathbf{F}}$] and [$\tilde{\mathbf{R}}$] are

$$\tilde{F}_{11} = -K_2K_3Q_2D_\rho^8 + [K_2(A_{33}Q_2 + Q_6 + K_3Q_3) - A_{23}^2Q_2\omega_2^2]D_\rho^6$$
$$+ Q_3(A_{23}^2\omega_2^2 - K_2A_{33})D_\rho^4 + C_1C_2(A_{33}K_2 - A_{23}^2\omega_2^2)D_\rho^2,$$
$$\tilde{F}_{12} = \tilde{F}_{21} = \omega_1\omega_2[A_{12}K_3Q_2D_\rho^8 + (A_{13}A_{23}Q_2 - A_{12}(A_{33}Q_2 + Q_6 + K_3Q_3))D_\rho^6$$
$$+ Q_3(A_{12}A_{33} - A_{13}A_{23})D_\rho^4 + C_1C_2(A_{13}A_{23} - A_{12}A_{33})D_\rho^2],$$
$$\tilde{F}_{13} = \tilde{F}_{31} = Q_9\omega_1[Q_2D_\rho^7 - Q_3D_\rho^5 + C_1C_2D_\rho^3],$$
$$\tilde{F}_{14} = \tilde{F}_{41} = -Q_9\omega_1^2[Q_5D_\rho^6 + C_1C_2D_\rho^4], \quad \tilde{F}_{15} = \tilde{F}_{51} = Q_9\omega_1\omega_2[Q_4D_\rho^6 - C_1C_2D_\rho^4],$$
$$\tilde{F}_{22} = -K_1K_3Q_2D_\rho^8 + [K_1(A_{33}Q_2 + Q_6 + K_3Q_3) - A_{13}^2Q_2\omega_1^2]D_\rho^6$$
$$+ Q_3(A_{13}^2\omega_1^2 - K_1A_{33})D_\rho^4 + C_1C_2(A_{33}K_1 - A_{13}^2\omega_1^2)D_\rho^2,$$
$$\tilde{F}_{23} = \tilde{F}_{32} = -Q_8\omega_2[Q_2D_\rho^7 - Q_3D_\rho^5 + C_1C_2D_\rho^3], \quad \tilde{F}_{24} = \tilde{F}_{42} = Q_8\omega_1\omega_2[Q_5D_\rho^6 + C_1C_2D_\rho^4],$$
$$\tilde{F}_{25} = \tilde{F}_{52} = -Q_8\omega_2^2[Q_4D_\rho^6 - C_1C_2D_\rho^4], \quad \tilde{F}_{33} = Q_1[Q_2D_\rho^8 - Q_3D_\rho^6 + C_1C_2D_\rho^4],$$
$$\tilde{F}_{34} = \tilde{F}_{43} = -Q_1\omega_1[Q_5D_\rho^7 + C_1C_2D_\rho^5], \quad \tilde{F}_{35} = \tilde{F}_{53} = Q_1\omega_2[Q_4D_\rho^7 - C_1C_2D_\rho^5],$$
$$\tilde{F}_{44} = -K_3K_5Q_1D_\rho^8 + [K_5(A_{33}Q_1 + Q_7) + C_2Q_1(K_3 - C_2\omega_2^2)]D_\rho^6 - C_2(A_{33}Q_1 + Q_7)D_\rho^4,$$
$$\tilde{F}_{45} = \tilde{F}_{54} = \omega_1\omega_2[A_{45}K_3Q_1D_\rho^8 + (C_1C_2Q_1 - A_{45}(A_{33}Q_1 + Q_7))D_\rho^6],$$
$$\tilde{F}_{55} = -K_3K_4Q_1D_\rho^8 + [K_4(A_{33}Q_1 + Q_7) + C_1Q_1(K_3 - C_1\omega_1^2)]D_\rho^6 - C_1(A_{33}Q_1 + Q_7)D_\rho^4,$$

and

$$\tilde{R}_{1j} = B_1[\omega_1\tilde{F}_{1j}D_\rho + v_2\omega_2\tilde{F}_{2j}D_\rho + (k_1 + v_2k_2)\tilde{F}_{3j}],$$
$$\tilde{R}_{2j} = B_2[v_1\omega_1\tilde{F}_{1j}D_\rho + \omega_2\tilde{F}_{2j}D_\rho + (k_2 + v_1k_1)\tilde{F}_{3j}],$$
$$\tilde{R}_{3j} = B_k[\omega_2\tilde{F}_{1j} + \omega_1\tilde{F}_{2j}]D_\rho, \quad \tilde{R}_{4j} = C_1[\omega_1\tilde{F}_{3j}D_\rho + \tilde{F}_{4j}],$$
$$\tilde{R}_{5j} = C_2[\omega_2\tilde{F}_{3j}D_\rho + \tilde{F}_{5j}], \quad \tilde{R}_{6j} = D_1[\omega_1\tilde{F}_{4j} + v_2\omega_2\tilde{F}_{5j}]D_\rho,$$
$$\tilde{R}_{7j} = D_2[v_1\omega_1\tilde{F}_{4j} + \omega_2\tilde{F}_{5j}]D_\rho, \quad \tilde{R}_{8j} = D_k[\omega_2\tilde{F}_{4j} + \omega_1\tilde{F}_{5j}]D_\rho \quad (j = 1, 2, \ldots, 5).$$

APPENDIX 2

For the equation

$$r^6 + \frac{a_1}{a_0} r^4 + \frac{a_2}{a_0} r^2 + \frac{a_3}{a_0} = 0, \tag{A1}$$

let

$$z = r^2, \quad A_i = a_i/a_0 \quad (i = 1, 2, 3). \tag{A2}$$

Then, eqn (A1) can be written as

$$z^3 + A_1 z^2 + A_2 z + A_3 = 0; \tag{A3}$$

this is a cubic algebraic equation and its solution can be discussed according to Abramowitz and Stegun (1966). Let

$$q = \tfrac{1}{3}A_2 - \tfrac{1}{9}A_1^2, \quad p = \tfrac{1}{6}(A_1 A_2 - 3A_3) - \tfrac{1}{27}A_1^3. \tag{A4}$$

Thus, when $q^3 + p^2 > 0$, eqn (A3) has one real root and a pair of complex conjugate roots; when $q^3 + p^2 = 0$, all roots are real and at least two are equal; and when $q^3 + p^2 < 0$, all roots are real.

Furthermore, let

$$s_1 = [p + (q^3 + p^2)^{1/2}]^{1/3}, \quad s_2 = [p - (q^3 + p^2)^{1/2}]^{1/3}; \tag{A5}$$

then the three roots of eqn (A3) can be written as

$$\begin{aligned} z_1 &= (s_1 + s_2) - \frac{A_1}{3}, \\ z_2 &= -\tfrac{1}{2}(s_1 + s_2) - \frac{A_1}{3} + \frac{i\sqrt{3}}{2}(s_1 - s_2), \\ z_3 &= -\tfrac{1}{2}(s_1 + s_2) - \frac{A_1}{3} - \frac{i\sqrt{3}}{2}(s_1 - s_2), \end{aligned} \tag{A6}$$

and

$$z_1 + z_2 + z_3 = -A_1, \quad z_1 z_2 + z_1 z_3 + z_2 z_3 = A_2, \quad z_1 z_2 z_3 = -A_3. \tag{A7}$$

Therefore, the solution of eqn (A1) is

$$r_1^2 = z_1, \quad r_2^2 = z_2, \quad r_3^2 = z_3. \tag{A8}$$